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On the Chromatic Number of General Kneser Hypergraphs

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Abstract

In a break-through paper, Lovász (1978) determined the chromatic number of Kneser graphs. This was improved by Schrijver (1978), by introducing the Schrijver subgraphs of Kneser graphs and showing that their chromatic number is the same as that of Kneser graphs. Alon, Frankl, and Lovász (1986) extended Lovász's result to the usual Kneser hypergraphs and one of our main results is to extend this to a new family of general Kneser hypergraphs. Moreover, as a special case, we settle a question from Naserasr and Tardif (2006).

In 2011, Meunier introduced almost 2-stable Kneser hypergraphs and determined their chromatic number as an approach to a supposition of Ziegler (2002) and a conjecture of Alon, Drewnowski, and Łuczak (2009). In this work, our second main result is to improve this by computing the chromatic number of a large family of Schrijver hypergraphs. Our last main result is to prove the existence of a completely multicolored complete bipartite graph in every coloring of a graph which extends a result of Simonyi and Tardos (2007).

The first two results are proved using a new improvement of the Dol'nikov-Kříž (1988, 1992) bound on the chromatic number of general Kneser hypergraphs.

Keywords: Chromatic Number, Kneser Hypergraphs, Z_p-Tucker Lemma, Tucker-Ky Fan's Lemma.
Subject classification: 05C15

1 Introduction

In this work, we derive a substantial improvement on the Dol'nikov-Kříž lower bound for the chromatic number of general Kneser hypergraphs (see Theorem A). This improvement is used in our determination of the chromatic number of some families of hypergraphs. The problem of finding a lower bound for the chromatic number of hypergraphs has been studied in the literature, see [7, 18, 19, 30, 31, 36, 37]. As in [25, 36], our proofs rely on the Z_p-Tucker lemma, which is a combinatorial generalization of the Borsuk-Ulam theorem. This lemma has also been used to simplify the

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evaluation of the chromatic number for some families of hypergraphs, e.g., almost 2-stable Kneser hypergraphs. Like the Borsuk-Ulam theorem, the Z_p-Tucker lemma is used to obtain lower bounds for the chromatic number of hypergraphs, see [25, 36].

First, in this section, we setup some notation and terminology. Hereafter, the symbol $[n]$ stands for the set $\{1,\ldots,n\}$. A *hypergraph* $\mathcal{F}$ is an ordered pair $(V(\mathcal{F}),E(\mathcal{F}))$, where $V(\mathcal{F})$ (the *vertex set*) is a finite set and $E(\mathcal{F})$ (the *hyperedge set*) is a family of distinct nonempty subsets of $V(\mathcal{F})$. If every hyperedge of a hypergraph has size r, then it is called an *r-uniform* hypergraph. A *t-coloring* of a hypergraph $\mathcal{F}$ is a mapping $h: V(\mathcal{F}) \longrightarrow [t] = \{1,2,\ldots,t\}$ such that no hyperedge is monochromatic. The minimum t such that there exists a t-coloring of the hypergraph $\mathcal{F}$ is called its *chromatic number*, and is denoted by $\chi(\mathcal{F})$. If $\mathcal{F}$ has a hyperedge of size 1, then we define the chromatic number of $\mathcal{F}$ to be infinite. Throughout this paper, for any hypergraph $\mathcal{F}$ with the vertex set $V(\mathcal{F}) = \{v_1,v_2,\ldots,v_n\}$, we consider a fixed bijective labeling $L_{\mathcal{F}}: [n] \longrightarrow V(\mathcal{F})$, i.e., a bijective map from $\{1,2,\ldots,n\}$ to $V(\mathcal{F})$. By abuse of notation, we assume that i and $L_{\mathcal{F}}(i)$ are referring to the same vertex of $\mathcal{F}$ and we use these representations interchangeably. The labeling $L_{\mathcal{F}}$ allows us to identify the vertex set of $\mathcal{F}$ with the set $[n]$. Let t be a positive integer and $N = (N_1,N_2,\ldots,N_t)$, where the N_i's are pairwise disjoint subsets of $V = [n]$. The *induced hypergraph* $\mathcal{F}_{|N}$ has $\bigcup\limits_{i=1}^{t} N_i$ and $\{A \in E(\mathcal{F}) : \ \exists i;\ 1 \le i \le t, A \subseteq N_i\}$ as vertex set and hyperedge set, respectively. For any hypergraph $\mathcal{F} = (V(\mathcal{F}),E(\mathcal{F}))$ and positive integer $r \ge 2$, the *general Kneser hypergraph* $\mathrm{KG}^r(\mathcal{F})$ is an r-uniform hypergraph whose vertex set is $E(\mathcal{F})$ and whose hyperedge set consists of all r-tuples of pairwise disjoint hyperedges of $\mathcal{F}$. In this terminology, the usual Kneser hypergraph is the hypergraph $\mathrm{KG}^r(\binom{[n]}{k})$, where $\binom{[n]}{k}$ denotes the hypergraph with vertex set $[n]$ and hyperedge set containing all k-subsets of $[n]$. A hypergraph $\mathcal{F}$ provides a *Kneser representation* for a graph G, if G and $\mathrm{KG}^2(\mathcal{F})$ are isomorphic. It is known that every graph has various Kneser representations; see [14] for more details.

A subset $S \subseteq [n]$ is *s-stable* (resp. *almost s-stable*) if any two distinct elements of S are at least "distance s apart" on the n-cycle (resp. n-path), that is, $s \le |i-j| \le n-s$ (resp. $|i-j| \ge s$) for distinct $i,j \in S$. Hereafter, for a subset $A \subseteq [n]$, the hypergraphs $\binom{A}{k}$, $\binom{A}{k}_s$, and $\binom{A}{k}_s^{\sim}$ have vertex set $[n]$ and edge sets consisting of all k-subsets of A, all s-stable k-subsets of A, and all almost s-stable k-subsets of A, respectively. One can see that $\binom{A}{k}_s \subseteq \binom{A}{k}_s^{\sim} \subseteq \binom{A}{k}$. Hereafter, for any positive integer $r \ge 2$, the hypergraphs $\mathrm{KG}^r(\binom{[n]}{k})$, $\mathrm{KG}^r(\binom{[n]}{k}_s)$, and $\mathrm{KG}^r(\binom{[n]}{k}_s^{\sim})$ are called the *"usual" Kneser hypergraph*, the *s-stable Kneser hypergraph*, and the *almost s-stable Kneser hypergraph*, and denoted by $\mathrm{KG}^r(n,k)$, $\mathrm{KG}^r(n,k)_{s-stab}$, and $\mathrm{KG}^r(n,k)_{s-stab}^{\sim}$, respectively.

In 1955, Kneser [17] conjectured that $\chi(\mathrm{KG}(n,k)) = n-2k+2$. In 1978, Lovász [20] used the Borsuk-Ulam theorem to determine the chromatic number of Kneser graphs. In 1973, Erdős [9] presented an upper bound for the chromatic number of the usual Kneser hypergraph $\mathrm{KG}^r(n,k)$ and conjectured that equality holds. In [2], this conjecture was confirmed and it was shown $\chi(\mathrm{KG}^r(n,k)) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. This implies that for any positive integer $r \ge 2$, we have $\chi(\mathrm{KG}^r(n,k)_{s-stab}) \le \chi(\mathrm{KG}^r(n,k)_{s-stab}^{\sim}) \le \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. In 2011, Meunier [25] improved this result to

$\chi(\mathrm{KG}^r(n,k)_{s-stab}) \leq \left\lceil \frac{n-s(k-1)}{r-1} \right\rceil$ for any $s \geq r$.

In 1988, Dol'nikov [7] generalized Lovász's result [17] and introduced a lower bound for the chromatic number of graphs. Next, Kříž [18] generalized Dol'nikov's bound to general Kneser hypergraphs. For a hypergraph $\mathcal{F}$, the r-*colorability defect* of $\mathcal{F}$, denoted $cd_r(\mathcal{F})$, is the minimum number of vertices which should be excluded so that the remaining subhypergraph is r-colorable, i.e.,

$$cd_r(\mathcal{F}) = \min\left\{|Y| \ : \ (V(\mathcal{F}) \setminus Y, \{F \in \mathcal{F} \ : \ F \cap Y = \varnothing\})\ is\ r-colorable\right\}.$$

Theorem A. (Dol'nikov for $r = 2$, Kříž [7, 18]) *For any hypergraph $\mathcal{F}$ and positive integer $r \geq 2$, we have*

$$\chi(\mathrm{KG}^r(\mathcal{F})) \geq \frac{cd_r(\mathcal{F})}{r-1}.$$

Consider the usual Kneser hypergraphs $\mathrm{KG}^r(n,k)$. In view of Theorem A, one can see that

$$\chi(\mathrm{KG}^r(n,k)) \geq \left\lceil \frac{cd_r(\binom{[n]}{k})}{r-1} \right\rceil \geq \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil.$$

To see this, let $Y \subseteq [n]$ such that the following hypergraph

$$\mathcal{H}_Y = ([n] \setminus Y, \{F \subseteq [n] \setminus Y \ : \ |F| = k\})$$

is r-colorable. This means that there exists a partition of the set $[n]\setminus Y$ into at most r subsets such that none of them contain any hyperedge of $\mathcal{H}_Y$; and consequently, of $\binom{[n]}{k}$. It follows that the size of any of these subsets is at most $k-1$; and therefore, the size of the set $[n]\setminus Y$ is at most $r(k-1)$. Hence, $cd_r(\binom{[n]}{k}) \geq n-r(k-1)$ and the assertion follows. In view of the prior discussion and the upper bound of Erdős [9] for the chromatic number of the usual Kneser hypergraph $\mathrm{KG}^r(n,k)$, one can see that $\chi(\mathrm{KG}^r(n,k)) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. Therefore, the Dol'nikov-Kříž theorem provides a tight lower bound for the chromatic number of the usual Kneser hypergraphs. In the present paper, we introduce the *alternation number* of hypergraphs as a generalization of the colorability defect and we present an improvement of the Dol'nikov-Kříž theorem.

In 2009, Alon et al. [3] constructed ideals in $\mathbb{N}$ which are not non-atomic while having the Nikodým property, by using stable Kneser hypergraphs. In this respect, they studied the chromatic number of the r-stable Kneser hypergraph $\mathrm{KG}^r(n,k)_{r-stab}$ and proved the following conjecture, provided that r is a power of 2. The following conjecture was presented as a supposition (2002) and a conjecture (2009) in [36] and [3], respectively.

Conjecture A. [3, 36] *Let $k, r,$ and n be positive integers, where $n \geq rk$ and $r \geq 2$. We have*

$$\chi(\mathrm{KG}^r(n,k)_{r-stab}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil.$$

As an approach to Conjecture A, Meunier [25] showed that $\chi(\mathrm{KG}^r(n,k)^{\sim}_{2-stab}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$ and he strengthened the above conjecture as follows.

Conjecture B. [25] *Let k, r, s, and n be positive integers, where $n \geq sk$ and $s \geq r \geq 2$. We have*

$$\chi(\mathrm{KG}^r(n,k)_{s-stab}) = \left\lceil \frac{n - s(k-1)}{r-1} \right\rceil.$$

In [16], it is shown that if n is sufficiently large and $r = 2$, then for $s \geq 4$, Conjecture B holds.

This paper is organized as follows. Section 2 is concerned with combinatorial topology and contains the main results of this paper. In Subsection 2.1, we introduce the i^{th} alternation number of hypergraphs as a combinatorial concept. Subsection 2.2 is devoted to Tucker's lemma and its generalizations. In Section 3, using the Z_p-Tucker lemma, we prove the main results of this paper. These results present some lower bounds for the chromatic number of general Kneser hypergraphs in terms of the i^{th} alternation number of hypergraphs, which improve Theorem A. In fact, the alternation number of a hypergraph can be considered as a generalization of colorability defect of hypergraphs. In Section 4, we present some applications of the aforementioned lower bounds for the chromatic number of hypergraphs. First, we determine the chromatic number of some families of multiple Kneser hypergraphs. In particular, we present a generalization of a result of Alon et al. [2] about the chromatic number of the usual Kneser hypergraphs. Furthermore, as a special case, we settle a question from Naserasr and Tardif [27]. Next, we extend Meunier's result and demonstrate that for any positive integers k, n, and r, if $n \not\equiv k \pmod{r-1}$ or r is an even integer, then for the *Schrijver hypergraph* $\mathrm{KG}^r(n,k)_{2-stab}$, we have $\chi(\mathrm{KG}^r(n,k)_{2-stab}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. Eventually, in Subsection 4.3, in view of Tucker-Ky Fan's lemma, we prove a multicolored-type result which confirms the existence of a completely multicolored complete bipartite graph in every coloring of a graph in terms of its altermatic number, which extends a result of Simonyi and Tardos [31].

2 Combinatorial Topology

In this section, we introduce some combinatorial topological tools. In the first subsection, we introduce the alternation number which owes its existence to Tucker's lemma, a combinatorial version of the Borsuk-Ulam theorem. We also present Tucker's lemma and some of its generalizations.

2.1 Alternation Number

Throughout, for any positive integer r, let $\mathcal{Z}_r = \{\omega_1, \omega_2, \ldots, \omega_r\}$ be a set of size r, where $0 \notin \mathcal{Z}_r$. Moreover, when p is a prime integer, we assume that $\mathcal{Z}_p$ is a *cyclic group* of order p and *generator* ω, i.e., $\mathcal{Z}_p = Z_p = \{\omega, \omega^2, \ldots, \omega^p\}$. In particular, when $r = 2$, we set $\mathcal{Z}_2 = Z_2 = \{\omega, \omega^2\} = \{-1, +1\}$.

One can consider $(\mathcal{Z}_r \cup \{0\})^n$ as the set of all *signed subsets* of $[n]$, that is, the family of all $(X^1, X^2, \ldots, X^r)$ of pairwise disjoint subsets of $[n]$. Precisely, for $X = (x_1, x_2, \ldots, x_n) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\}$ and $j \in [r]$, we define $X^j = \{i \in [n] : x_i = \omega_j\}$. For any $X \in (\mathcal{Z}_r \cup \{0\})^n$, we use these representations interchangeably, i.e., $X = (x_1, x_2, \ldots, x_n)$ or $X = (X^1, X^2, \ldots, X^r)$. Let $X, Y \in (\mathcal{Z}_r \cup \{0\})^n$. By $X \preceq Y$, we mean $X^j \subseteq Y^j$ for each $j \in [r]$.

Consider a *ground set* $\mathcal{S}$ such that $0 \notin \mathcal{S}$ and $|\mathcal{S}| \geq 2$. Let $(z_1, z_2, \ldots, z_n)$ be a sequence of elements of $\mathcal{S}$. The subsequence $z_{a_1}, z_{a_2}, \ldots, z_{a_t}$ $(1 \leq a_1 < a_2 < \cdots < a_t \leq n)$ is said to be an *alternating subsequence* if any two consecutive terms in this subsequence are different. Hereafter, by abuse of language, by an alternating subsequence of $X = (x_1, x_2, \ldots, x_n) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\}$, we mean an alternating subsequence of nonzero terms of X. We denote by $alt(X)$ the length of a longest alternating subsequence of nonzero terms of X. Moreover, define $alt((0, 0, \ldots, 0)) = 0$ and denote the number of nonzero terms of X by $|X|$. For instance, if $\mathcal{S} = \mathcal{Z}_4$ and $X = (\omega_4, 0, \omega_2, \omega_1, 0, \omega_1, \omega_3, \omega_1)$, then $alt(X) = 5$ and $|X| = 6$. Note that if $X, Y \in (\mathcal{Z}_r \cup \{0\})^n$ and $X \preceq Y$, then every alternating subsequence of X is also an alternating subsequence of Y, and therefore, $alt(X) \leq alt(Y)$. Also, note that if the first nonzero term in X is ω_j, then every alternating subsequence of X of maximum length begins with ω_j, implying that X^j contains the smallest integer in $\bigcup\limits_{\ell=1}^{r} X^\ell$. For a permutation σ of $[n]$, we denote by $alt_\sigma(X)$ the length of a longest alternating subsequence of $(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$, i.e., $alt((x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}))$. In this terminology, $alt(X)$ is the same as $alt_I(X)$, where I is the identity permutation.

Let $\mathcal{F}$ be a hypergraph with n vertices. By abuse of notation and under a fixed bijective labeling $L_\mathcal{F} : [n] \longrightarrow V(\mathcal{F})$, we may identify $V(\mathcal{F})$ with $[n]$. Let i, r, and n be positive integers, where $r \geq 2$. Also, let σ be a permutation of $[n]$ (a *total ordering* of $[n]$). Set $alt_{r,\sigma}(\mathcal{F}, i)$ to be the largest integer t such that there exists an $X = (X^1, X^2, \ldots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\}$ with $alt_\sigma(X) = t$ and that the chromatic number of the hypergraph $\mathrm{KG}^r(\mathcal{F}_{|X})$ is at most $i - 1$. Note that the hypergraph $\mathcal{F}_{|X}$ has $\bigcup\limits_{i=1}^{r} X^i$ and $\{A \in E(\mathcal{F}) :\ \exists i;\ 1 \leq i \leq r, A \subseteq X^i\}$ as vertex set and hyperedge set, respectively.

In particular, $alt_{r,\sigma}(\mathcal{F}, 1)$ is the largest integer t such that there exists an $X = (X^1, X^2, \ldots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\}$ with $alt_\sigma(X) = t$ and none of the X^j's contain any hyperedge of $\mathcal{F}$. If every singleton is a hyperedge, then we define $alt_{r,\sigma}(\mathcal{F}, 1) = 0$. Also, if $\mathcal{F}$ is an empty hypergraph, i.e., $E(\mathcal{F}) = \varnothing$, then $alt_{r,\sigma}(\mathcal{F}, i) = |V(\mathcal{F})|$. Hereafter, $alt_{r,\sigma}(\mathcal{F}, 1)$ is denoted by $alt_{r,\sigma}(\mathcal{F})$. Now set $alt_r(\mathcal{F}, i) = \min\{alt_{r,\sigma}(\mathcal{F}, i);\ \sigma \in S_n\}$. Also, $alt_r(\mathcal{F}, i)$ is called the i^{th} *alternation number* of $\mathcal{F}$ (with respect to r) and the *first alternation number* of $\mathcal{F}$ is denoted by $alt_r(\mathcal{F})$. In this terminology, if $i > \chi(\mathrm{KG}^r(\mathcal{F}))$, then $alt_r(\mathcal{F}, i) = n$. For instance, one can see that for $n \geq rk$, $alt_{r,I}(\binom{[n]}{k}_2) = r(k-1) + 1$.

Here we introduce the main results of this paper. Indeed, we present two lower bounds for the chromatic number of general Kneser hypergraphs in terms of the alternation number of hypergraphs. Section 3 is devoted to the proof of these results.

Theorem 1. *Let $\mathcal{F}$ be a hypergraph and p be a prime number. For any positive integer i, where $i \leq \chi(\mathrm{KG}^p(\mathcal{F})) + 1$, we have*

$$\chi(\mathrm{KG}^p(\mathcal{F})) \geq \frac{|V(\mathcal{F})| - alt_p(\mathcal{F}, i)}{p - 1} + i - 1.$$

Theorem 1 motivates us to define the i^{th} *altermatic number* of graphs which provides a lower bound for the chromatic number of graphs. A *graph homomorphism* from a graph G to a graph H is a mapping $f : V(G) \longrightarrow V(H)$ which preserves the adjacency. We use the symbol $G \longleftrightarrow H$ to denote that each graph admits a homomorphism to the other. For any graph G and positive integer $i \in \{1, 2, \ldots, \chi(G)+1\}$, define the i^{th} altermatic number $\zeta(G, i)$ of G as follows

$$\zeta(G, i) = \max_{\mathcal{F}}\{|V(\mathcal{F})| - alt(\mathcal{F}, i) + i - 1 : \mathrm{KG}^2(\mathcal{F}) \longleftrightarrow G\}.$$

For simplicity, when $i = 1$, $\zeta(G, i)$ is denoted by $\zeta(G)$. Now in view of Theorem 1, we have the following result.

Theorem 2. *For any graph G and for any $i \in \{1, 2, \ldots, \chi(G) + 1\}$, we have*

$$\chi(G) \geq \zeta(G, i).$$

Moreover, we generalize Theorem 1 to any arbitrary positive integer r for the case $i = 1$.

Theorem 3. *For any hypergraph $\mathcal{F}$ and positive integer $r \geq 2$, we have*

$$\chi(\mathrm{KG}^r(\mathcal{F})) \geq \frac{|V(\mathcal{F})| - alt_r(\mathcal{F})}{r - 1}.$$

Remark: Theorem 3 is an improvement of the Dol'nikov-Kříž theorem. Define $M_r(\mathcal{F})$ to be the maximum size of a set $T \subseteq V(\mathcal{F})$ such that the induced hypergraph of $\mathcal{F}$ on T,

$$\mathcal{F}_{|_T} = (T, \{F \in E(\mathcal{F}) \ : \ F \subseteq T\}),$$

is r-colorable. One can see that $cd_r(\mathcal{F}) = n - M_r(\mathcal{F})$. In view of Theorem A, we know $\chi(\mathrm{KG}^r(\mathcal{F})) \geq \frac{cd_r(\mathcal{F})}{r-1} = \frac{|V(\mathcal{F})|-M_r(\mathcal{F})}{r-1}$. One can check that for any permutation σ of $[n]$, $M_r(\mathcal{F}) \geq alt_{r,\sigma}(\mathcal{F})$. To see this, in view of the definition of $alt_{r,\sigma}(\mathcal{F})$, there is an $X \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\}$, such that none of the X^j's $(1 \leq j \leq r)$ contain any hyperedge of $\mathcal{F}$ and that $alt_\sigma(X) = alt_{r,\sigma}(\mathcal{F})$. Set $T = \cup_{j=1}^r X^j$. Obviously, $(X^1, \ldots, X^r)$ is a proper r-coloring of $\mathcal{F}_{|_T}$. This implies $|X| \leq M_r(\mathcal{F})$. On the other hand, $alt_\sigma(X) \leq |X|$. Therefore,

$$\chi(\mathrm{KG}^r(\mathcal{F})) \geq \frac{|V(\mathcal{F})| - alt_\sigma(X)}{r - 1} \geq \frac{|V(\mathcal{F})| - M_r(\mathcal{F})}{r - 1} = \frac{cd_r(\mathcal{F})}{r - 1}.$$

Theorem 3 is in general better than the Dol'nikov-Kříž lower bound. Ziegler in [36, 37] showed $cd_r(\binom{[n]}{k}_t) = \max\{n - tr(k-1), 0\}$. Therefore, the Dol'nikov-Kříž theorem implies $\chi(\mathrm{KG}^r(n, k)_{2-stab}) \geq \frac{\max\{n-2r(k-1),0\}}{r-1}$. Although, one can easily see that for $n \geq rk$, $alt_{r,I}(\binom{[n]}{k}_2) = r(k-1) + 1$; and so by Theorem 3,

$$\chi(\mathrm{KG}^r(n, k)_{2-stab}) \geq \frac{n - r(k-1) - 1}{r - 1} > \frac{\max\{n - 2r(k-1), 0\}}{r - 1}.$$

If r is a prime number, then Theorem 1 is stronger than Theorem 3. For example, one can see that for $n \geq 2k$, $alt_{2,I}(\binom{[n]}{k}_2, 1) = alt_{2,I}(\binom{[n]}{k}_2, 2) = 2k - 1$. Therefore, Theorem 3 implies that $\chi(\mathrm{KG}^2(n, k)_{2-stab}) \geq n - 2k + 1$ while in view of Theorem 1 and for $i = 2$, one can obtain the well-known result of Schrijver [28], i.e., $\chi(\mathrm{KG}^2(n, k)_{2-stab}) = \zeta(\mathrm{KG}^2(n, k)_{2-stab}, 2) = n - 2k + 2$.

2.2 Tucker's Lemma and Its Generalizations

In this subsection, we present Tucker's lemma and some of its generalizations. For more details about the Borsuk-Ulam theorem and Tucker's lemma, we refer the reader to [21].

Lemma A. (Tucker's lemma [33]) *Let n be a positive integer and $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\} \longrightarrow \{\pm1, \pm2, \ldots, \pm(n-1)\}$. Also, assume that for any signed set $X \in \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\}$, we have $\lambda(-X) = -\lambda(X)$. Then there exist two signed sets X and Y such that $X \preceq Y$ and also $\lambda(X) = -\lambda(Y)$.*

There are several interesting and surprising applications of Tucker's lemma in combinatorics, including a combinatorial proof of Lovász-Kneser's theorem by Matoušek [22]. There are also various generalizations of Tucker's lemma. The next lemma is a combinatorial variant of the Z_p-Tucker lemma proved and modified in [36] and [25], respectively.

Lemma B. (Z_p-Tucker Lemma) *Let m, n, p, and α be nonnegative integers, where $m, n \geq 1$, $m \geq \alpha \geq 0$, and p is prime. Let*

$$\begin{aligned} \lambda :\ (Z_p \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\} &\longrightarrow Z_p \times [m] \\ X &\longmapsto (\lambda_1(X), \lambda_2(X)) \end{aligned}$$

be a map satisfying the following properties:

1. *λ is a Z_p-equivariant map, that is, for each $\omega^j \in Z_p$, we have $\lambda(\omega^j X) = (\omega^j \lambda_1(X), \lambda_2(X))$.*

2. *for all $X_1 \preceq X_2 \in (Z_p \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\}$, if $\lambda_2(X_1) = \lambda_2(X_2) \leq \alpha$, then $\lambda_1(X_1) = \lambda_1(X_2)$.*

3. *for all $X_1 \preceq X_2 \preceq \cdots \preceq X_p \in (Z_p \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\}$, if $\lambda_2(X_1) = \lambda_2(X_2) = \cdots = \lambda_2(X_p) \geq \alpha + 1$, then the $\lambda_1(X_i)$'s are not pairwise distinct for $i = 1, 2, \ldots, p$.*

Then $\alpha + (m - \alpha)(p - 1) \geq n$.

Another interesting generalization of the Borsuk-Ulam theorem is Ky Fan's lemma [10]. This lemma has been used in several articles to study some coloring properties of graphs, see [5, 12] .

Lemma C. (Tucker-Ky Fan's lemma [10]) *Let m and n be positive integers and $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\} \longrightarrow \{\pm1, \pm2, \ldots, \pm m\}$ be a map satisfying the following properties:*

1. *for any $X \in \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\}$, we have $\lambda(-X) = -\lambda(X)$ (a Z_2-equivariant map),*

2. *no two signed sets X and Y are such that $X \preceq Y$ and $\lambda(X) = -\lambda(Y)$.*

Then there are n signed sets $X_1 \preceq X_2 \preceq \cdots \preceq X_n$ such that $\{\lambda(X_1), \ldots, \lambda(X_n)\} = \{+c_1, -c_2, \ldots, (-1)^{n-1} c_n\}$, where $1 \leq c_1 < \cdots < c_n \leq m$. In particular, $m \geq n$.

3 Proofs of the Main Results

This section is devoted to the proofs of Theorem 1 and Theorem 3. Let h be a proper coloring of $\mathrm{KG}^r(\mathcal{F})$ with colors $\{1,2,\ldots,C\}$. For any subset $B \subseteq V(\mathcal{F}) = [n]$, we define $\bar{h}(B) = \max\{h(S) : S \subseteq B, S \in E(\mathcal{F})\}$. If B contains no hyperedge of $\mathcal{F}$, then set $\bar{h}(B) = 0$. For any $X = (X^1, X^2, \ldots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$, define $\bar{h}(X) = \max\{\bar{h}(X^1), \bar{h}(X^2), \ldots, \bar{h}(X^r)\}$. Now we are ready to prove Theorem 1.

Proof of Theorem 1. Without loss of generality, we assume that $V(\mathcal{F}) = [n]$. Consider an arbitrary total ordering $\leqslant$ on $2^{[n]}$. To prove the assertion, it is sufficient to show that for any $\sigma \in S_n$, $\chi(\mathrm{KG}^p(\mathcal{F})) \geq \frac{n - alt_{p,\sigma}(\mathcal{F},i)}{p-1} + i - 1$. Without loss of generality, assume $\sigma = I$. Let $\mathrm{KG}^p(\mathcal{F})$ be properly colored with C colors $\{1,2,\ldots,C\}$. For any $F \in E(\mathcal{F})$, we denote its color by $h(F)$. Set $\alpha = alt_{p,I}(\mathcal{F},i)$ and $m = alt_{p,I}(\mathcal{F},i) + C - i + 1$.

Now define a map $\lambda : (Z_p \cup \{0\})^n \setminus \{(0,0,\ldots,0)\} \longrightarrow Z_p \times [m]$ as follows

- If $X \in (\mathcal{Z}_p \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ and $alt_I(X) \leq alt_{p,I}(\mathcal{F},i)$, set $\lambda(X) = (\omega^j, alt_I(X))$, where j is the index of the set X^j containing the smallest integer in $\bigcup\limits_{j=1}^{p} X^j$ (ω^j is then the first nonzero term in $X = (x_1,\ldots,x_n)$).
- If $X \in (\mathcal{Z}_p \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ and $alt_I(X) \geq alt_{p,I}(\mathcal{F},i) + 1$, in view of definition of $alt_{p,I}(\mathcal{F},i)$, the chromatic number of $\mathrm{KG}^p(\mathcal{F}_{|X})$ is at least i. Set $\lambda(X) = (\omega^j, \bar{h}(X) - i + 1 + \alpha)$, where j is a positive integer such that there is an $A \in E(\mathcal{F})$ where $A \subseteq X^j$, $h(A) = \bar{h}(X)$, and that A is the largest such a subset with respect to the total ordering $\leqslant$ (note that $\bar{h}(X) \geq i$).

One can check that λ is a Z_p-equivariant map from $(Z_p \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ to $Z_p \times [m]$.

Let $X_1 \preceq X_2 \in (Z_p \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$. If $\lambda_2(X_1) = \lambda_2(X_2) \leq \alpha$, then the length of a longest alternating subsequence of X_1 is the same as that of X_2. Therefore, the first nonzero terms of X_1 and X_2 are equal, and equivalently, $\lambda_1(X_1) = \lambda_1(X_2)$.

Let $X_1 \preceq X_2 \preceq \cdots \preceq X_p \in (Z_p \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ such that $\lambda_2(X_1) = \lambda_2(X_2) = \cdots = \lambda_2(X_p) \geq \alpha + 1$. According to the definition of λ, for each $1 \leq a \leq p$, there are $F_a \in E(\mathcal{F})$ and $j_a \in [p]$ such that $F_a \subseteq X_a^{j_a}$ and $\lambda_2(X_a) = h(F_a) - i + 1 + \alpha$. This implies $h(F_1) = h(F_2) = \cdots = h(F_p)$. If $|\{j_1, j_2, \ldots, j_p\}| = p$, then $\{F_1, F_2, \ldots, F_p\}$ is a hyperedge in $\mathrm{KG}^p(\mathcal{F})$ and this contradicts the assumption that h is a proper coloring.

Now we can apply the Z_p-Tucker lemma and conclude that $n \leq alt_{p,I}(\mathcal{F},i) + (C - i + 1)(p-1)$, which yields $C \geq \frac{n - alt_{p,I}(\mathcal{F},i)}{p-1} + i - 1$. ■

For any $\sigma \in S_n$, $N = \{a_1, a_2, \ldots, a_l\} \subseteq [n]$, and $X \in (\mathcal{Z}_r \cup \{0\})^n$, define $alt_{\sigma_{|N}}(X)$ to be $alt(x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_l)})$ such that $i_1 < i_2 < \cdots < i_l$ and $N = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_l)\}$. Also, let $alt_{r,\sigma_{|N}}(\mathcal{F}_{|N})$ be the largest integer t such that there is an $X \in (\mathcal{Z}_r \cup \{0\})^n$ with $alt_{\sigma_{|N}}(X) = t$ and none of the X^j's $(1 \leq j \leq r)$ contain any hyperedge of $\mathcal{F}_{|N}$. Also, if $N = \varnothing$, then set $alt_{r,\sigma_{|N}}(\mathcal{F}_{|N}) = 0$. In view of Theorem 1, for any prime number p, $\chi(\mathrm{KG}^p(\mathcal{F}_{|N})) \geq \frac{|N| - alt_{r,\sigma_{|N}}(\mathcal{F}_{|N})}{p-1}$.

Lemma 1. *Let $r, s,$ and n be positive integers, $X = (X^1, X^2, \ldots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\}$, and σ be a permutation of $[n]$. Also, assume that for each $j \in [r]$, $Y^{1j}, Y^{2j}, \ldots, Y^{sj}$ are pairwise disjoint subsets of X^j. If we set*

$$Y_j = (Y^{1j}, Y^{2j}, \ldots, Y^{sj}) \in (\mathcal{Z}_s \cup \{0\})^n$$

and

$$Z = \left(Y^{11}, \ldots, Y^{s1}, \ldots, Y^{1r}, \ldots, Y^{sr}\right) \in (\mathcal{Z}_{rs} \cup \{0\})^n \setminus \{(0, 0, \ldots, 0)\},$$

then $alt_\sigma(Z) \geq \sum_{j=1}^{r} alt_{\sigma|_{X^j}}(Y_j)$.

Proof. Without loss of generality, assume $\sigma = I$. Also, let $alt_I(X) = t$. If $x_{a_1}, x_{a_2}, \ldots, x_{a_t}$ form an alternating subsequence of X ($1 \leq a_1 < a_2 < \cdots < a_t \leq n$), then the set $\{a_1, a_2, \ldots, a_t\}$ is called the *index set* of this alternating subsequence. Choose an alternating subsequence $x_{a_1}, x_{a_2}, \ldots, x_{a_t}$ of X such that a_1 is the smallest integer in $T = \cup_{j=1}^{r} X^j$ and that for each $i \in [t]$, there is a $j_i \in [r]$ where $[a_i, a_{i+1}) \cap T \subseteq X^{j_i}$. For each $j \in [r]$, assume that P_j is a longest alternating subsequence of Y_j. Now we present an alternating subsequence P of Z. Construct P such that for each $i \in [t]$, P and P_{j_i} have the same index set in $[a_i, a_{i+1})$. It is straightforward to check that P is an alternating subsequence of Z and also $|P| = \sum_{j=1}^{r} alt_{I|_{X^j}}(Y_j)$. ■

Here, we extend Theorem 1 to r-uniform hypergraphs (r is not necessarily prime) for the first alternation number.

Lemma 2. *Let r and s be positive integers, where $r, s \geq 2$. Also, assume that for any hypergraph $\mathcal{H}$, $\chi(\mathrm{KG}^r(\mathcal{H})) \geq \frac{|V(\mathcal{H})| - alt_r(\mathcal{H})}{r-1}$ and $\chi(\mathrm{KG}^s(\mathcal{H})) \geq \frac{|V(\mathcal{H})| - alt_s(\mathcal{H})}{s-1}$. For any hypergraph $\mathcal{F}$, we have $\chi(\mathrm{KG}^{rs}(\mathcal{F})) \geq \frac{|V(\mathcal{F})| - alt_{rs}(\mathcal{F})}{rs-1}$.*

Proof. Let $\mathcal{F}$ be a hypergraph with $V(\mathcal{F}) = [n]$. It is enough to show that for any $\sigma \in S_n$, $\chi(\mathrm{KG}^{rs}(\mathcal{F})) \geq \frac{n - alt_{rs,\sigma}(\mathcal{F})}{rs-1}$. Without loss of generality, suppose $\sigma = I$. Let $C = \chi(\mathrm{KG}^{rs}(\mathcal{F}))$. Now suppose, contrary to the assertion, that

$$n - alt_{rs,I}(\mathcal{F}) > (rs - 1)C. \tag{1}$$

Define the hypergraph $\mathcal{T}$ as

$$V(\mathcal{T}) = [n] \quad \& \quad E(\mathcal{T}) = \left\{N \subseteq [n] : \ |N| - alt_{s,I|_N}(\mathcal{F}_{|N}) > (s-1)C\right\}.$$

Now according to the assumption and the definition of $\mathcal{T}$, for each $N \in E(\mathcal{T})$, we have

$$(s-1)\chi(\mathrm{KG}^s(\mathcal{F}_{|N})) \geq |N| - alt_{s,I|_N}(\mathcal{F}_{|N}) > (s-1)C.$$

Consequently,

$$\chi(\mathrm{KG}^s(\mathcal{F}_{|N})) > C. \tag{2}$$

In the following claim, we prove $n - alt_{r,I}(\mathcal{T}) > (r-1)C$ which also implies $E(\mathcal{T}) \neq \varnothing$.
Claim: $n - alt_{r,I}(\mathcal{T}) > (r-1)C$.

Suppose, contrary to the claim, that $n - alt_{r,I}(\mathcal{T}) \leq (r-1)C$, and therefore, $alt_{r,I}(\mathcal{T}) \geq n-(r-1)C$. Hence, in view of the definition of $alt_{r,I}(\mathcal{T})$, there exists an $X = (X^1, X^2, \ldots, X^r) \in (\mathcal{Z}_r \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ such that $alt_I(X) \geq n-(r-1)C$ and none of the X^j's contain any hyperedge of $\mathcal{T}$. In particular, none of them lie in $E(\mathcal{T})$. Note that if $E(\mathcal{T}) = \varnothing$, then we set X to be a vector in $(\mathcal{Z}_r \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ with $alt(X) \geq n-(r-1)C$. Therefore, by the definition of $\mathcal{T}$, for any $1 \leq j \leq r$

$$|X^j| - alt_{s,I_{|_{X^j}}}(\mathcal{F}_{|_{X^j}}) \leq (s-1)C,$$

and consequently, $|X^j| - (s-1)C \leq alt_{s,I_{|_{X^j}}}(\mathcal{F}_{|_{X^j}})$. Hence, for each $j \in [r]$, there are s pairwise disjoint sets $Y^{1j}, \ldots, Y^{sj} \subseteq X^j$, such that $alt_{I_{|_{X^j}}}(Y^{1j}, \ldots, Y^{sj}) \geq |X^j| - (s-1)C$ and none of them contain any hyperedge of $\mathcal{F}_{|_{X^j}}$. Set

$$Z = \left(Y^{11}, \ldots, Y^{s1}, \ldots, Y^{1r}, \ldots, Y^{sr}\right) \in (\mathcal{Z}_{rs} \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}.$$

By Lemma 1,

$$alt_I(Z) \geq \sum_{j=1}^{r} alt_{I_{|_{X^j}}}(Y^{1j}, \ldots, Y^{sj}) \geq \sum_{j=1}^{r}(|X^j| - (s-1)C) \geq \left(\sum_{j=1}^{r}|X^j|\right) - r(s-1)C.$$

Note that $\sum_{j=1}^{r}|X^j| \geq alt_I(X)$, thus

$$alt_I(Z) \geq alt_I(X) - r(s-1)C \geq n-(r-1)C - r(s-1)C = n-(sr-1)C.$$

Since none of the Y^{ij}'s contain any hyperedge of $\mathcal{F}$, we obtain $alt_{rs,I}(\mathcal{F}) \geq n-(sr-1)C$ which contradicts inequality (1). So the claim follows.

By the assumption and the claim, we have

$$(r-1)\chi(\mathrm{KG}^r(\mathcal{T})) \geq n - alt_{r,I}(\mathcal{T}) > (r-1)C,$$

and so

$$\chi(\mathrm{KG}^r(\mathcal{T})) > C. \tag{3}$$

Now consider a proper coloring $h : \mathcal{F} \longrightarrow [C]$ of $\mathrm{KG}^{rs}(\mathcal{F})$. By inequality (2), for any $N \in E(\mathcal{T})$, there exists a color in $[C]$ which is assigned to s pairwise disjoint members of $\mathcal{F}_{|_N}$. Now define $h' : \mathcal{T} \longrightarrow [C]$ such that $h'(N)$ is the maximum color which h assigns this color to s pairwise disjoint sets in $\mathcal{F}_{|_N}$. According to inequality (3), there are r pairwise disjoint sets $N_1, \ldots, N_r \in E(\mathcal{T})$ such that h' assigns them the same color. Thus there are rs pairwise disjoint sets $F_{ik} \in E(\mathcal{F})$ such that for any $1 \leq i \leq r$ and $1 \leq k \leq s$, $F_{ik} \subseteq N_i$, and that h assigns them the same color. This contradicts our assumption. ■

Now we are in a position to prove Theorem 3.

Proof of Theorem 3. The proof is a direct consequence of Theorem 1 (the case $i = 1$) and Lemma 2. ■

4 Applications

This section is divided to three parts. In the first subsection, we use Theorem 2 to determine the chromatic number of a family of hypergraphs. The next subsection contains some result concerning the chromatic number of stable Kneser hypergraphs. Finally, in the last subsection, we study the existence of multicolored subgraphs in a coloring of a graph.

4.1 Multiple Kneser Hypergraphs

Throughout this part, let k, r, m, and n be positive integers where $r \geq 2$ and $k \geq 1$. Furthermore, let $\pi = (P_1, P_2, \ldots, P_m)$ be a *partition* of $[n]$ and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive integer vector. The *multiple Kneser hypergraph* $\mathrm{KG}^r(\pi; \vec{s}; k)$ is a hypergraph with the vertex set

$$V = \{A : \ A \subseteq P_1 \cup P_2 \cup \cdots \cup P_m, \ |A| = k, \forall 1 \leq i \leq m; \ |A \cap P_i| \leq s_i\},$$

where $\{A_1, \ldots, A_r\}$ is a hyperedge if $A_1, A_2, \ldots, A_r$ are pairwise disjoint. In what follows, we determine the chromatic number of the multiple Kneser hypergraphs, provided that $r = 2$ or $|P_i| \leq 2s_i$, for any $1 \leq i \leq m$. For this purpose, we define the function $f_{r,\pi}$ as follows

$$f_{r,\pi}(P_i) = \begin{cases} rs_i & \text{if } |P_i| \geq rs_i \\ \\ |P_i| & \text{otherwise.} \end{cases}$$

Also, set

$$M_{r,\pi} = \max\left\{ rk - 1 + \sum_{j=1}^{t} (|P_{i_j}| - f_{r,\pi}(P_{i_j})) \ : \ t \in [m] \ \& \ \sum_{j=1}^{t} f_{r,\pi}(P_{i_j}) \leq rk - 1 \right\},$$

where $i_1, \ldots, i_t$ are distinct elements of $\{1, 2, \ldots, m\}$ and the maximum is taken over all possible values for them. In what follows, we first introduce the main results of this subsection (Theorem 4 and Theorem 5) and derive some applications. These results are going to be proved in the remaining of this subsection.

Theorem 4. *Let k, m, and n be positive integers with $k \geq 1$. Also, let $\pi = (P_1, P_2, \ldots, P_m)$ be a partition of $[n]$ and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive integer vector. If $\mathrm{KG}^2(\pi; \vec{s}; k)$ is a nonempty graph, then*

$$\chi(\mathrm{KG}^2(\pi; \vec{s}; k)) = \zeta(\mathrm{KG}^2(\pi; \vec{s}; k)) = n - M_{2,\pi} + 1.$$

Note that Theorem 4 provides a generalization of Lovász-Kneser theorem [20]. In fact, if we set $|P_1| = |P_2| = \cdots = |P_m| = 1$ and $s_1 = s_2 = \cdots = s_m = 1$, then $\mathrm{KG}^2(\pi; \vec{s}; k) = \mathrm{KG}^2(m, k)$.

In [32], Tardif introduced the graph $K_t^{k,m}$ and called it the *fractional multiple* of the complete graph K_t. This graph can be represented as follows. The vertices of $K_t^{k,m}$ represent the independent sets of size k in a disjoint union of m copies of K_t, and two of these are joined by an edge in $K_t^{k,m}$ if they are disjoint. In [26, 27], it was shown that $\chi(K_t^{k,m}) = t(m-k+1)$, where t is an even integer and $k \le m$. However, determining the chromatic number of $K_t^{k,m}$ for any odd integer $t \ge 3$ remained an open problem. Moreover, it was conjectured in [26] that $\chi(K_t^{k,m}) = t(m-k+1)$, where $t \ge 3$ is odd and $k \le m$. Note that $K_{2t+3}^{k,m}$ contains a complete join of $K_{2t}^{k,m}$ and $K_3^{k,m}$. Hence, if we show $\chi(K_3^{k,m}) = 3(m-k+1)$, then the aforementioned conjecture holds; for more details see [27].

If we set $|P_1| = |P_2| = \cdots = |P_m| = t$ and $s_1 = s_2 = \cdots = s_m = 1$, then $\mathrm{KG}^2(\pi; \vec{s}; k) = K_t^{k,m}$. Therefore, in view of Theorem 4, we have the next corollary which gives an affirmative answer to the aforementioned conjecture [26].

Corollary 1. *If t, k, and m are positive integers, where $k \le m$ and $t \ge 2$, then $\chi(K_t^{k,m}) = t(m-k+1)$.*

Alon et al. [2] determined the chromatic number of the usual Kneser hypergraphs. Precisely, it was shown $\chi(\mathrm{KG}^r(n,k)) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. Here, we introduce a generalization of their result.

Theorem 5. *Let k, r, m, and n be positive integers, where $r \ge 2$, $k \ge 1$, and $n \ge rk$. Also, let $\pi = (P_1, P_2, \ldots, P_m)$ be a partition of $[n]$ and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive integer vector, where for each $i \in [m]$, $|P_i| \le 2s_i$. We have*

$$\chi(\mathrm{KG}^r(\pi; \vec{s}; k)) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

Note that if $\sum_{j=1}^{m} \min\{|P_j|,\ s_j\} < k$, then $\mathrm{KG}^r(\pi; \vec{s}; k)$ has no vertex, and therefore, $\chi(\mathrm{KG}^r(\pi; \vec{s}; k)) = 0$. Also, one can check that the hypergraph $\mathrm{KG}^r(\pi; \vec{s}; k)$ is nonempty (that is, $E(\mathrm{KG}^r(\pi; \vec{s}; k)) \ne \varnothing$) if and only if $\sum_{j=1}^{m} f_{r,\pi}(P_j) \ge kr$. To this end, it is easy to check that if $E(\mathrm{KG}^r(\pi; \vec{s}; k)) \ne \varnothing$, then $\sum_{j=1}^{m} f_{r,\pi}(P_j) \ge kr$. Now let $\sum_{j=1}^{m} f_{r,\pi}(P_j) \ge kr$. For $j = 1, 2, \ldots, m$, suppose that P'_j is a subset of P_j such that $|P'_j| = f_{r,\pi}(P_j)$. Set $Y = \bigcup_{j=1}^{m} P'_j$. Without loss of generality, assume that $Y = \{1, 2, \ldots, n'\}$ and each P'_j is made of consecutive integers. For any $1 \le i \le r$, define $A_i = \{a \in Y :\ a \le rk \ \&\ a \overset{r}{\equiv} i\}$. One can check that $\{A_1, A_2, \ldots, A_r\}$ forms a hyperedge of $\mathrm{KG}^r(\pi; \vec{s}; k)$, which implies $E(\mathrm{KG}^r(\pi; \vec{s}; k)) \ne \varnothing$.

In the next lemma, we give an upper bound for the chromatic number of the multiple Kneser hypergraph $\mathrm{KG}^r(\pi; \vec{s}; k)$ in terms of n and $M_{r,\pi}$.

Lemma 3. *Let k, r, m, and n be positive integers where $r \ge 2$ and $k \ge 1$. Also, let $\pi = (P_1, P_2, \ldots, P_m)$ be a partition of $[n]$ and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive* *integer vector. We have*

$$\chi(\mathrm{KG}^r(\pi;\vec{s};k)) \le \max\left\{1, \left\lceil \frac{n-M_{r,\pi}}{r-1}+1\right\rceil\right\}.$$

Proof. If $\chi(\mathrm{KG}^r(\pi;\vec{s};k)) \le 1$, then there is nothing to prove. Now assume that $\mathrm{KG}^r(\pi;\vec{s};k)$ is a nonempty hypergraph. This implies $\sum\limits_{j=1}^{m} f_{r,\pi}(P_j) \ge kr$; and consequently, $M_{r,\pi} < n$. Without loss of generality, suppose that t is the largest positive integer such that the value of $M_{r,\pi}$ is attained. Moreover, suppose that $M_{r,\pi}$ is obtained by $P_m, P_{m-1}, \ldots, P_{m-t+1}$, i.e., $M_{r,\pi} = rk-1+\sum_{j=m-t+1}^{m}(|P_j| - f_{r,\pi}(P_j))$ and $\sum_{j=m-t+1}^{m} f_{r,\pi}(P_j) \le rk-1$. By the definition of $M_{r,\pi}$ and because of the maximality of t, for any $1 \le i \le m-t$, one can see that $rk - \sum\limits_{j=m-t+1}^{m} f_{r,\pi}(P_j) \le f_{r,\pi}(P_i)$. If $m-t=0$, then $\sum\limits_{j=1}^{m} f_{r,\pi}(P_j) \le rk-1$ which is impossible. Hence, suppose $m-t \ge 1$. Consider L to be a subset of P_{m-t} of size $rk-1-\sum\limits_{j=m-t+1}^{m} f_{r,\pi}(P_j)$. Set $T = L \cup \left(\bigcup\limits_{j=m-t+1}^{m} P_j\right)$. Now we present a proper coloring for $\mathrm{KG}^r(\pi;\vec{s};k)$ using $\left\lceil \frac{n-M_{r,\pi}}{r-1}\right\rceil + 1$ colors. To this end, it will be shown that all the vertices of $\mathrm{KG}^r(\pi;\vec{s};k)$ which are subsets of T form an independent set. On the contrary, suppose that $A_1, A_2, \ldots, A_r \in V(\mathrm{KG}^r(\pi;\vec{s};k))$ form a hyperedge in $\mathrm{KG}^r(\pi;\vec{s};k)$ where $A_1, A_2, \ldots, A_r \subseteq T$. According to the definition of $\mathrm{KG}^r(\pi;\vec{s};k)$, we have $\sum\limits_{i=1}^{r} |A_i \cap P_j| \le f_{r,\pi}(P_j)$ for any $m-t+1 \le j \le m$ and also $\sum\limits_{i=1}^{r} |A_i \cap L| \le |L|$. Thus

$$\begin{aligned}
rk = |\bigcup_{i=1}^{r} A_i| &= (\sum_{i=1}^{r} |A_i \cap L|) + \sum_{j=m-t+1}^{m} \sum_{i=1}^{r} |A_i \cap P_j| \\
&\le |L| + \sum_{j=m-t+1}^{m} f_{r,\pi}(P_j) \\
&= (kr-1) - \sum_{j=m-t+1}^{m} f_{r,\pi}(P_j) + \sum_{j=m-t+1}^{m} f_{r,\pi}(P_j)
\end{aligned}$$

which is a contradiction.

Note that the size of $D = (\bigcup\limits_{j=1}^{m-t} P_j) \setminus L$ is $n - M_{r,\pi}$ and $T \cup D = [n]$. Set $b = \left\lceil \frac{n-M_{r,\pi}}{r-1}\right\rceil$. Consider a partition $(Q_1, Q_2, \ldots, Q_b)$ of D such that $r-1 = |Q_1| = |Q_2| = \cdots = |Q_{b-1}| \ge |Q_b| > 0$.

Now we present a proper coloring for $\mathrm{KG}^r(\pi;\vec{s};k)$ using $b+1$ colors. As mentioned, all the vertices of $\mathrm{KG}^r(\pi;\vec{s};k)$ which are subsets of T form an independent set; and therefore, we can assign a color to all of them, e.g., $b+1$. Since every other vertex A has a nonempty intersection with D, we define the color of this vertex to be the minimum integer j such that $A \cap Q_j \neq \varnothing$. ∎

In what follows, we show $alt_2(V(\mathrm{KG}^2(\pi;\vec{s};k))) = M_{2,\pi} - 1$.

Lemma 4. *Let k, m, and n be positive integers and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive integer vector. Also, let $\pi = (P_1, P_2, \ldots, P_m)$ be a partition of $[n]$, where each P_i is a subset of $|P_i|$ consecutive numbers. If $X \in \{+1, 0, -1\}^n \setminus \{(0,0,\ldots,0)\}$ and $alt_I(X) \geq M_{2,\pi}$, then X^+ or X^- contains a k-subset A of $[n]$ such that A is a vertex of $\mathrm{KG}^2(\pi;\vec{s};k)$.*

Proof. It is sufficient to show that if $X \in \{+1, 0, -1\}^n \setminus \{(0,0,\ldots,0)\}$ and $alt_I(X) = |X| = M_{2,\pi}$, then the assertion holds. Let $X \in \{+1,0,-1\}^n \setminus \{(0,0,\ldots,0)\}$ with $alt_I(X) = |X| = M_{2,\pi}$. Note that in this case, any two consecutive nonzero terms of X have different signs. Therefore, the sequence of nonzero terms of X is the longest alternating subsequence of X. Define $alt_I(X, P_j)$ to be the length of the longest alternating subsequence of X lying in P_j. For a contradiction, suppose that neither X^+ nor X^- contains a k-subset A of $[n]$ such that A is a vertex of $\mathrm{KG}^2(\pi;\vec{s};k)$. Set $I(X) = \sum\limits_{j=1}^{m} \min\{f_{2,\pi}(P_j),\ alt_I(X,P_j)\}$. One can see that

$$\begin{aligned} 2k-2 &\geq \sum_{j=1}^{m} \min\{f_{2,\pi}(P_j),\ alt_I(X,P_j)\} \\ &= \sum_{\{j:\ alt_I(X,P_j) \geq f_{2,\pi}(P_j)\}} f_{2,\pi}(P_j) + \sum_{\{j:\ alt_I(X,P_j) < f_{2,\pi}(P_j)\}} alt_I(X,P_j). \end{aligned}$$

This means that $\sum\limits_{\{j:\ alt_I(X,P_j) \geq f_{2,\pi}(P_j)\}} f_{2,\pi}(P_j) \leq 2k-2$, and according to the definition of $M_{2,\pi}$, we have

$$2k - 1 + \sum_{\{j:\ alt_I(X,P_j) \geq f_{2,\pi}(P_j)\}} (|P_j| - f_{2,\pi}(P_j)) \leq M_{2,\pi}.$$

Hence,

$$2k - 1 - M_{2,\pi} + \sum_{\{j:\ alt_I(X,P_j) \geq f_{2,\pi}(P_j)\}} |P_j| \leq \sum_{\{j:\ alt_I(X,P_j) \geq f_{2,\pi}(P_j)\}} f_{2,\pi}(P_j).$$

Now we have

$$2k - 1 - M_{2,\pi} + \sum_{\{j:\ alt_I(X,P_j) \geq f_{2,\pi}(P_j)\}} |P_j| + \sum_{\{j:\ alt_I(X,P_j) < f_{2,\pi}(P_j)\}} alt_I(X,P_j) \leq I(X).$$

On the other hand, $I(X) \leq 2k-2$; and so

$$1 + alt_I(X) \leq 1 + \sum_{\{j:\ alt_I(X,P_j) \geq f_{2,\pi}(P_j)\}} |P_j| + \sum_{\{j:\ alt_I(X,P_j) < f_{2,\pi}(P_j)\}} alt_I(X,P_j) \leq M_{2,\pi},$$

which is a contradiction. ∎

Proof of Theorem 4. Without loss of generality, suppose that each P_i is made of consecutive integers. Since $\mathrm{KG}^2(\pi;\vec{s};k)$ is a nonempty graph, $\sum\limits_{j=1}^{m} f_{2,\pi}(P_j)\geq 2k$, and thus, $M_{2,\pi}<n$. To this end, without loss of generality, suppose that t is the largest positive integer such that the value of $M_{2,\pi}$ is attained. Moreover, suppose that $M_{2,\pi}$ is obtained by $P_m,P_{m-1},\ldots,P_{m-t+1}$, i.e., $M_{2,\pi}=2k-1+\sum_{j=m-t+1}^{m}(|P_j|-f_{2,\pi}(P_j))$ and $\sum_{j=m-t+1}^{m} f_{2,\pi}(P_j)\leq 2k-1$. Note that $\sum\limits_{j=1}^{m} f_{2,\pi}(P_j)\geq 2k$. Therefore, by the definition of $M_{2,\pi}$, we have $1\leq m-t$. Also, because of the maximality of t, for any $1\leq i\leq m-t$, one can see that

$$2k-1-\sum_{j=m-t+1}^{m} f_{2,\pi}(P_j)\leq f_{2,\pi}(P_i)-1<|P_i|.$$

This implies

$$M_{2,\pi}=2k-1+\sum_{j=m-t+1}^{m}(|P_j|-f_{2,\pi}(P_j))<\sum_{j=m-t}^{m}|P_j|\leq n.$$

Set $\mathcal{F}=V(\mathrm{KG}^2(\pi;\vec{s};k))$. In view of Lemma 4, $alt_{2,I}(\mathcal{F})\leq M_{2,\pi}-1$. Consequently, by Theorem 2 and Lemma 3,

$$n-M_{2,\pi}+1\geq\chi(\mathrm{KG}^2(\pi;\vec{s};k))\geq\zeta(\mathrm{KG}^2(\pi;\vec{s};k))\geq n-alt_{2,I}(\mathcal{F})\geq n-(M_{2,\pi}-1),$$

which completes the proof. ■

Proof of Theorem 5. Without loss of generality, suppose that each P_i is made of consecutive integers. Note that since $n=\sum\limits_{j=1}^{m} f_{r,\pi}(P_j)\geq rk$, $\mathrm{KG}^r(\pi;\vec{s};k)$ is a nonempty hypergraph. One can check that $M_{r,\pi}=rk-1$. Set $\mathcal{F}=V(\mathrm{KG}^r(\pi;\vec{s};k))$. In view of Theorem 3 and Lemma 3, the proof is completed by showing that $alt_{r,I}(\mathcal{F})\leq M_{r,\pi}-r+1=r(k-1)$. It is sufficient to show that if $X\in(\mathcal{Z}_r\cup\{0\})^n\setminus\{(0,0,\ldots,0)\}$ and $alt_I(X)=|X|=r(k-1)+1$, then there exists some $j\in[r]$ such that X^j contains some members of $\mathcal{F}$. Let $X=(X^1,X^2,\ldots,X^r)\in(\mathcal{Z}_r\cup\{0\})^n\setminus\{(0,0,\ldots,0)\}$ be a vector such that $alt_I(X)=|X|=r(k-1)+1$. Define $alt_I(X,P_j)$ to be the length of the longest alternating subsequence of X lying in P_j. Also, for each $\omega_i\in\mathcal{Z}_r$, let $alt(X,P_j,\omega_i)$ be the number of ω_i's of the longest alternating subsequence of X lying in P_j. One can see that $alt_I(X,P_j)=\sum\limits_{i=1}^{r} alt(X,P_j,\omega_i)$. It is then enough to prove that there exists some $\omega_i\in\mathcal{Z}_r$ such that

$$\sum_{j=1}^{m}\min\{s_j,alt(X,P_j,\omega_i)\}\geq k.$$

On the contrary, suppose

$$\sum_{i=1}^{r}\sum_{j=1}^{m}\min\{s_j,alt(X,P_j,\omega_i)\}\leq r(k-1).$$

Note that for any $j \in [m]$, $|P_j| \le 2s_j$. This implies that for every $j \in [m]$ and $\omega_i \in \mathcal{Z}_r$, $\min\{s_j, alt(X, P_j, \omega_i)\} = alt(X, P_j, \omega_i)$. Consequently,

$$alt_I(X) = \sum_{j=1}^{m}\sum_{i=1}^{r} \min\{s_j, alt(X, P_j, \omega_i)\} \le r(k-1),$$

which is a contradiction. ∎

4.2 Stable Kneser Hypergraphs

In this part, we investigate the chromatic number of s-stable Kneser hypergraphs and almost s-stable Kneser hypergraphs. The next proposition was proved in [25] and here we present another proof of this result.

Proposition 1. *Let $k, n, r,$ and s be nonnegative integers, where $n \ge sk$ and $s \ge r \ge 2$. We have*

$$\chi(\mathrm{KG}^r(n,k)_{s-stab}) \le \left\lceil \frac{n - s(k-1)}{r-1} \right\rceil.$$

Proof. Let $n = sq + t$, where $0 < t \le s$. Set $P_{i+1} = \{is+1, is+2, \ldots, (i+1)s\}$ for $0 \le i \le q-1$ and $P_{q+1} = \{sq+1, sq+2, \ldots, n\}$. Also, define $\pi = (P_1, P_2, \ldots, P_{q+1})$ and $\vec{s} = (1, 1, \ldots, 1)$. By Lemma 3,

$$\chi(\mathrm{KG}^r(\pi; \vec{s}; k)) \le \left\lceil \frac{n - M_{r,\pi}}{r-1} + 1 \right\rceil.$$

One can check that $M_{r,\pi} = (k-1)s + r - 1$. Since $\mathrm{KG}^r(n,k)_{s-stab}$ is a subgraph of $\mathrm{KG}^r(\pi; \vec{s}, k)$, one obtains

$$\chi(\mathrm{KG}^r(n,k)_{s-stab}) \le \left\lceil \frac{n - s(k-1)}{r-1} \right\rceil,$$

which completes the proof. ∎

Lemma 5. *Let r and s be positive integers, where $r \ge s \ge 2$. Assume that for any $n \ge rk$, $\chi(\mathrm{KG}^r(n,k)_{s-stab}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. For any prime number p and any $n \ge prk$, we have $\chi(\mathrm{KG}^{pr}(n,k)_{s-stab}) = \left\lceil \frac{n-pr(k-1)}{pr-1} \right\rceil$.*

Proof. We endow $2^{[n]}$ with an arbitrary total ordering $\preccurlyeq$. Set $C = \left\lceil \frac{n-pr(k-1)}{pr-1} \right\rceil$. According to Proposition 1, $\chi(\mathrm{KG}^{pr}(n,k)_{s-stab}) \le C = \left\lceil \frac{n-pr(k-1)}{pr-1} \right\rceil$. For a contradiction, suppose that there is an integer $n \ge prk$ such that $\chi(\mathrm{KG}^{pr}(n,k)_{s-stab}) < C$. Let h be a proper $(C-1)$-coloring of $\mathrm{KG}^{pr}(n,k)_{s-stab}$. For any subset $A \subseteq [n]$, where $|A| \ge rk$, define the hypergraph $\mathcal{F}_A = (A, \binom{A}{k}_s)$. One can see that the hypergraph $\mathrm{KG}^r(|A|,k)_{s-stab}$ can be considered as a subhypergraph of $\mathrm{KG}^r(\mathcal{F}_A)$. Hence, by the assumption, we have

$$\chi(\mathrm{KG}^r(\mathcal{F}_A)) \ge \chi(\mathrm{KG}^r(|A|,k)_{s-stab}) = \left\lceil \frac{|A| - r(k-1)}{r-1} \right\rceil.$$

Consequently, if $|A| > (r-1)(C-1) + r(k-1)$, then $\chi(\mathrm{KG}^r(\mathcal{F}_A)) > C-1$. In this case, there are r pairwise disjoint vertices of $\mathrm{KG}^r(\mathcal{F}_A)$ such that h assigns the same color to all of them. Set $m = p\left((r-1)(C-1) + r(k-1)\right) + C - 1$. Now we introduce a map $\lambda: (Z_p \cup \{0\})^n \setminus \{(0,0,\ldots,0)\} \longrightarrow Z_p \times [m]$. First, note that if $X \in (Z_p \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ and $alt_I(X) > p\left((r-1)(C-1) + r(k-1)\right)$, then there is an $1 \le i \le p$ such that $|X^i| > (r-1)(C-1) + r(k-1)$. So, $\chi(\mathrm{KG}^r(\mathcal{F}_{X^i})) > C-1$; and therefore, there are r pairwise disjoint vertices $B_1, B_2, \ldots, B_r$ of $\mathrm{KG}^r(\mathcal{F}_{X^i})$ such that $h(B_1) = \cdots = h(B_r) = c$. Set $\bar{h}(X)$ to be the largest such color c. Precisely,

$$\bar{h}(X) = \max\left\{c : \exists i, B_1, \ldots, B_r \in \binom{X^i}{k}_s, B_j \cap B_l = \varnothing, h(B_1) = \cdots = h(B_r) = c\right\}.$$

Now define $\lambda(X)$ as follows

- if $alt_I(X) \le p\left((r-1)(C-1) + r(k-1)\right)$, set $\lambda(X) = (\omega^j, alt_I(X))$ such that j is the smallest integer in $\bigcup\limits_{k=1}^{p} X^k$.
- if $alt_I(X) \ge p((r-1)(C-1)+r(k-1))+1$, define $\lambda(X) = (\omega^i, p(r-1)(C-1) + pr(k-1) + \bar{h}(X))$ such that there are r pairwise disjoint vertices $B_1, B_2, \ldots, B_r$ for which $h(B_1) = h(B_2) = \cdots = h(B_r) = \bar{h}(X)$, $\bigcup\limits_{k=1}^{r} B_k \subset X^i$ and X^i is the largest component of $X = (X^1, X^2, \ldots, X^p)$ with respect to the ordering $\preccurlyeq$.

One can see that λ satisfies the conditions of the Z_p-Tucker lemma; and therefore, we should have

$$D = p(r-1)(C-1) + pr(k-1) + (C-1)(p-1) \ge n.$$

On the other hand,

$$\begin{array}{rcl} D & = & (pr-1)(C-1) + pr(k-1) \\ & \le & (pr-1)\left(\frac{n-pr(k-1)+pr-2}{pr-1} - 1\right) + pr(k-1) \\ & = & n-1, \end{array}$$

which is a contradiction. ■

It was proved in [3] that for $r = 2^j$, $\chi(\mathrm{KG}^r(n,k)_{r-stab}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. The next corollary is an immediate consequence of this result and Lemma 5.

Corollary 2. *Let $a, k, n,$ and r be positive integers, where $n \ge rk$. If $2^a | r$, then $\chi(\mathrm{KG}^r(n,k)_{2^a-stab}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$.*

Theorem 6. *For any positive integers $k, n,$ and r with $n \ge rk$, if $n \not\equiv k \pmod{r-1}$ or r is an even integer, then $\chi(\mathrm{KG}^r(n,k)_{2-stab}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$.*

Proof. By Corollary 2, if r is an even integer, then there is nothing to prove. Note that $\mathrm{KG}^r(n,k)_{2-stab}$ is a subgraph of $\mathrm{KG}^r(n,k)$. This implies $\chi(\mathrm{KG}^r(n,k)_{2-stab}) \le \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$. Hence, it is sufficient to show that if $n \not\equiv k \pmod{r-1}$, then we have $\chi(\mathrm{KG}^r(n,k)_{2-stab}) \ge \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$.

Let $X \in (Z_r \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ and $alt_I(X) \ge r(k-1)+2$. One can see that there exists at least one X^j (for some $1 \le j \le r$) containing some vertex of $\mathrm{KG}^r(n,k)_{2-stab}$. Therefore, $alt_{r,I}(\binom{[n]}{k}_2) \le r(k-1)+1$. By Theorem 3,

$$\chi(\mathrm{KG}^r(n,k)_{2-stab}) \ge \left\lceil \frac{n - alt_{r,I}(\binom{[n]}{k}_2)}{r-1} \right\rceil \ge \left\lceil \frac{n-r(k-1)-1}{r-1} \right\rceil.$$

Also, $\left\lceil \frac{n-r(k-1)-1}{r-1} \right\rceil = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$, provided that $n \not\equiv k \pmod{r-1}$. ■

In view of the Z_p-Tucker lemma, Meunier [25] proved that, for any positive integer r and any $n \ge kp$, the chromatic number of $\mathrm{KG}^r(n,k)^{\sim}_{2-stab}$ is the same as the chromatic number of $\mathrm{KG}^r(n,k)$, namely that is equal to $\left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$.

Theorem B. [25] *For positive integers k, r, and n, if $r \ge 2$ and $n \ge rk$, then* $\chi(\mathrm{KG}^r(n,k)^{\sim}_{2-stab}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil$.

Proof. We proceed in the same fashion as in the proof of Theorem 6. If $X \in (Z_r \cup \{0\})^n \setminus \{(0,0,\ldots,0)\}$ and $alt_I(X) \ge r(k-1)+1$, then there exists at least an X^j (for some $1 \le j \le r$) containing some vertex of $\mathrm{KG}^r(n,k)^{\sim}_{2-stab}$. Therefore, $alt_{r,I}(\binom{[n]}{k}^{\sim}_2) \le r(k-1)$. By Theorem 3,

$$\chi(\mathrm{KG}^r(n,k)^{\sim}_{2-stab}) \ge \left\lceil \frac{n - alt_{r,I}(\binom{[n]}{k}^{\sim}_2)}{r-1} \right\rceil \ge \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil,$$

and the result follows. ■

4.3 Multicolored Graphs

We say that a graph is *completely multicolored* in a coloring if all its vertices receive different colors.

Theorem C. [31] *Let $\mathcal{F}$ be a hypergraph and $d = cd_2(\mathcal{F})$. For every proper coloring of $\mathrm{KG}^2(\mathcal{F})$ with colors $\{1,2,\ldots,C\}$ (C arbitrary), there exists a completely multicolored complete bipartite subgraph $K_{\lceil \frac{d}{2} \rceil, \lfloor \frac{d}{2} \rfloor}$ of $\mathrm{KG}^2(\mathcal{F})$ such that the d different colors alternate on the two sides of the bipartite subgraph with respect to their natural order.*

We should mention that there are several versions of Theorem C in terms of other topological parameters, see [6, 11, 29, 30]. The latter theorem presents a lower bound for *local chromatic number* of a graph which is the minimum number of colors that must appear within distance 1 of a vertex, for more about the local chromatic number, see [8, 30]. The next theorem provides a generalization of Theorem C in terms of the altermatic number of graphs.

Theorem 7. *Let G be a graph. For every proper coloring of G with colors $\{1, 2, \ldots, C\}$ (C arbitrary), there exists a completely multicolored complete bipartite subgraph $K_{\lceil \frac{\zeta(G)}{2} \rceil, \lfloor \frac{\zeta(G)}{2} \rfloor}$ of G such that the $\zeta(G)$ different colors occur alternating on the two sides of the bipartite subgraph with respect to their natural order.*

Proof. Consider a hypergraph $\mathcal{F}$ such that $\mathrm{KG}^2(\mathcal{F})$ is isomorphic to G and that $\zeta(G) = |V(\mathcal{F})| - alt_2(\mathcal{F})$. Let $V(\mathcal{F}) = [n]$. Without loss of generality, suppose $t = alt_2(\mathcal{F}) = alt_{2,I}(\mathcal{F})$, where I is the identity permutation on $[n]$. Let h be a proper coloring of $\mathrm{KG}^2(\mathcal{F})$ with C colors $\{1, 2, \ldots, C\}$. Now we construct a map $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \ldots, 0)\} \longrightarrow \{\pm 1, \pm 2, \ldots, \pm m\}$, where $m = t + C$. For $X \in \{-, 0, +\}^n \setminus \{(0, 0, \ldots, 0)\}$, set $\lambda(X)$ as follows

- If $alt_I(X) \leq alt_{2,I}(\mathcal{F})$, define $\lambda(X) = \pm alt_I(X)$, where the sign is determined by the sign of the first element (with respect to the permutation I) of a longest alternating subsequence of X (which is actually the first nonzero term of X).

- If $alt_I(X) \geq alt_{2,I}(\mathcal{F}) + 1$, in view of definition of $alt_I(\mathcal{F})$, X^+ or X^- contains a hyperedge of $\mathcal{F}$. Define $\bar{h}(X) = \max\{h(F) : F \in E(\mathcal{F}_{|X})\}$. Assume that F is a hyperedge of $\mathcal{F}_{|X}$ such that $h(F) = \bar{h}(X)$. Set $\lambda(X) = \pm(h(F) + t)$, where the sign is positive if $F \subseteq X^+$ and negative if $F \subseteq X^-$.

It is straightforward to see that $\lambda : \{-, 0, +\}^n \setminus \{(0, 0, \ldots, 0)\} \longrightarrow \{\pm 1, \pm 2, \ldots, \pm m\}$ satisfies the conditions of Tucker-Ky Fan's lemma. Therefore, by Tucker-Ky Fan's lemma, there are n signed sets $X_1 \preceq X_2 \preceq \cdots \preceq X_n$ such that $\{\lambda(X_1), \ldots, \lambda(X_n)\} = \{c_1, -c_2, c_3, \ldots, (-1)^{n-1} c_n\}$ where $1 \leq c_1 < c_2 < \cdots < c_n \leq m$.

Since $1 \leq |X_1| < |X_2| < \cdots < |X_n| \leq n$, we have $|X_i| = i$. Note that $|\lambda|$ is a monotone function; and thus, $\lambda(X_i) = (-1)^{i-1} c_i$. This implies $|X_i^+| = \lceil \frac{i}{2} \rceil$ and $|X_i^-| = \lfloor \frac{i}{2} \rfloor$. In particular, $|X_t^+| = \lceil \frac{t}{2} \rceil$ and $|X_t^-| = \lfloor \frac{t}{2} \rfloor$.

Note that for $i \geq t + 1$, $|\lambda(X_i)| = \bar{h}(X_i) + t$ which implies that

- if i is even, then $\bar{h}(X_i^-) = c_i - t$
- if i is odd, then $\bar{h}(X_i^+) = c_i - t$.

Now for any even integer $i \in \{t + 1, t + 2, \ldots, n\}$, there is an $F_i \in E(\mathcal{F})$ such that $F_i \subseteq X_i^- \subseteq X_n^-$ and $h(F_i) = c_i - t$. Also, for any odd integer $i \in \{t + 1, t + 2, \ldots, n\}$, there is a $G_i \in E(\mathcal{F})$ such that $G_i \subseteq X_i^+ \subseteq X_n^+$ and $h(G_i) = c_i - t$. Set $A = \{F_i : t + 1 \leq i \leq n \text{ and } i \text{ is even}\}$ and $B = \{G_i : t + 1 \leq i \leq n \text{ and } i \text{ is odd}\}$. Since $X_n^+ \cap X_n^- = \varnothing$, the bipartite subgraph $\mathrm{KG}^2(\mathcal{F})[A, B]$ is the desired complete bipartite subgraph. ■

Let p and q be positive integers, where $p \geq 2q$. For any graph G, a (p, q)-*coloring* of G is a mapping $h : V(G) \longrightarrow \{0, 1, \ldots, p - 1\}$ such that for any edge $xy \in E(G)$, we have $q \leq |h(x) - h(y)| \leq p - q$. The *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf \left\{ \frac{p}{q} : G \text{ admits a } (p, q) - \text{coloring} \right\}$$

It is well-known that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. For more on the circular chromatic number, see [34, 35].

The problem whether a graph has the same chromatic number and circular chromatic number has received attention, see [1, 24, 30, 34, 35]. For a t-coloring h of G, a cycle $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is called *tight* if $h(v_{i+1}) \stackrel{t}{\equiv} h(v_i) + 1$ for $i = 0, 1, \ldots, n-1$, where the indices of the vertices are modulo n. It is known [34] that for a positive integer t, $\chi_c(G) = t$ if and only if G is t-colorable and every t-coloring of G has a tight cycle.

For a graph G, by Theorem 7, one can see that if $\chi(G) = \zeta(G)$, then for any $\chi(G)$-coloring of G, there is a multicolored complete bipartite subgraph $K_{\lceil \frac{\chi(G)}{2} \rceil, \lfloor \frac{\chi(G)}{2} \rfloor}$ of G. This result implies that $\chi_c(G) = \chi(G)$, provided that $\chi(G)$ is an even integer.

Corollary 3. *Let G be a graph and $\chi(G) = \zeta(G)$. If $\chi(G)$ is an even integer, then $\chi_c(G) = \chi(G)$.*

It was conjectured in [15] that every Kneser graph has the same chromatic number and circular chromatic number. This conjecture has been studied in several papers, see [1, 4, 5, 13, 15, 24, 30]. Finally, Chen [5] completely proved this conjecture by using Tucker-Ky Fan's lemma in an innovative way. Next, a shorter proof was presented in [4]. The next corollary is a consequence of Theorem 4 and Corollary 3.

Corollary 4. *Let k, n, and m be positive integers, where $k \geq 1$. Also, let $\pi = (P_1, P_2, \ldots, P_m)$ be a partition of $[n]$ and $\vec{s} = (s_1, s_2, \ldots, s_m)$ be a positive integer vector such that $\mathrm{KG}^2(\pi; \vec{s}; k)$ is a nonempty graph. If $\chi(\mathrm{KG}^2(\pi; \vec{s}; k))$ is an even integer, then $\chi_c(\mathrm{KG}^2(\pi; \vec{s}; k)) = \chi(\mathrm{KG}^2(\pi; \vec{s}; k))$.*

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